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VARIABLE VISCOSITY EFFECTS IN SEVERAL NATURAL CONVECTION FLOWS

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Abstract—A regular perturbation analysis is presented for three laminar natural convection flows in liquids with temperature dependent viscosity: a freely-rising plane plume, the flow above a horizontal line source on an adiabatic surface (a plane wall plume) and the flow adjacent to a vertical uniform flux surface. While these flows have well-known power-law similarity solutions when the fluid viscosity is taken to be constant, they are non-similar when the viscosity is considered to be a function of temperature. A single similar flow, that adjacent to a vertical isothermal surface, is also analyzed for comparison in order to estimate the extent of validity of the perturbation analysis. The formulation used here provides a unified treatment of variable viscosity effects on these four flows. With the exception of water, the major temperature variation of the fluid properties of common liquids is seen to be in the absolute viscosity. This has been previously recognized and utilized for other flows and is the basis for the applicability of the present analysis. Computed first-order perturbation quantities are presented for all four flows. Several interesting variable viscosity trends on flow and transport are suggested by the present results. These modifications to a constant viscosity formulation are seen to be significant even within the necessarily limited range of a first-order perturbation analysis. Heat transfer results for the isothermal and uniform heat flux surfaces are in very close agreement with the corresponding data and correlations of previous investigations. The present results also place some previous conclusions regarding plume flows in clearer perspective.

NOMENCLATURE

a_n, coefficients in equation (6b);
b,c,d, defined in equations (5a)–(5d);
c_p, specific heat of fluid;
f, non-dimensional stream function;
Gr_x, local Grashof number, $= g\beta\rho^2 x^3(t_0 - t_\infty)/\mu_f^2$;
Gr_x^*, local flux Grashof number, $= g\beta q'' x^4/k\nu^2$;
g, acceleration due to gravity;
H, heat-transfer parameter, $= (Nu_x)_\infty/(Gr_x)_\infty^{1/4}$;
H^*, flux heat-transfer parameter, $= (Nu_x)_\infty/(Gr_x^*)_\infty^{1/5}$;
h, local heat-transfer coefficient;
k, thermal conductivity of fluid;
M, momentum flux in the x direction;
$\dot{m}$, mass flow rate per unit width of surface;
N,n, defined in equations (5a)–(5d);
Nu_x, local Nusselt number, $= hx/k$;
N', heat-transfer parameter, $= \sqrt{2}Nu_x/(Gr_x')^{1/4}$;
Q, total heat convected downstream;
q'', surface heat flux;
t, temperature;
t_e, reference temperature, $= t_0 - 1/4(t_0 - t_\infty)$;
t_f, film temperature, $= (t_0 + t_\infty)/2$;
u, vertical velocity component;
v, horizontal velocity component;
x, vertical coordinate;
y, horizontal coordinate.

Greek symbols

β, coefficient of thermal expansion;
γ_f, viscosity parameter, $= \left(\frac{1}{\mu}\frac{d\mu}{dt}\right)_f (t_0 - t_\infty)$;
γ_f^*, viscosity parameter, $= \left(\frac{1}{\mu}\frac{d\mu}{dt}\right)_f (t_0 - t_\infty)_0$;
ε_H, perturbation quantity, $= (t_0 - t_\infty)/t_\infty$;
η, non-dimensional horizontal distance;
μ, absolute viscosity of fluid;
ν, kinematic viscosity of fluid;
ρ, density;
σ, Prandtl number of fluid;
ϕ, temperature excess ratio, $= (t - t_\infty)/(t_0 - t_\infty)$;
ψ, stream function;
τ, shear stress.

Subscripts

CM, refers to results from Carey and Mollendorf [31];
F, refers to results from Fujii *et al.* [22];
f, refers to conditions at film temperature;
m, refers to conditions at mean temperature;
w, refers to conditions at the wall;
o, refers to conditions at $x = 0$;
∞, refers to conditions in the ambient fluid;
0, refers to conditions when $\gamma_f^* = 0$.

INTRODUCTION

DEPARTURES from constant-fluid-property descriptions of convective transport at moderate and high temperature differences have been evident for some time. A brief discussion of variable property treatment in forced-flow is presented here first, since many of the physical considerations are similar to those of natural convection. For flow in internal passages, the radial temperature distribution is known to modify the velocity profile through temperature–viscosity coupling. As a result, higher viscosity near the surface tends to reduce transport. The viscosity of gases generally increases with temperature, whereas liquid viscosities decrease with temperature. Therefore, for heating a fluid, the effect of temperature-dependent viscosity is to decrease transport in gases and to increase transport in liquids. The opposite occurs for cooling a fluid. Sieder and Tate [1] proposed to correlate this effect with the ratio of the absolute viscosities at the average (inlet and exit) and wall temperatures raised to the 0.14 power, i.e. $(\mu_m/\mu_w)^{0.14}$. Using this factor, all other fluid properties were evaluated at t_m.

The earliest analysis of variable property effects was done by Schuh [2] for the external forced-flow of air and oils over a flat plate. A later analysis by Cohen and Reshotko [3] considered a linear variation of viscosity as well as compressibility effects for various pressure gradients in the external flow of an ideal gas. Later, Seban [4] extended the Prandtl number and viscosity ratio ranges used by Schuh [2]. Poots and Raggett [5] and [6] have analyzed the external forced-flow of water over a flat plate, rotating disk and circular cylinder. For each configuration, the effects of variable density, viscosity, specific heat and thermal conductivity were included.

Both measurement and analysis were done by Test [7] and Hwang and Hong [8] to assess the effect of variable viscosity on heat transfer in SAE60 oil in a tube and ethylene glycol in a rectangular duct. Hwang and Hong [8] considered both isothermal and constant heat flux boundary conditions, and used an inverse variation of viscosity with temperature, as did Schuh [2] and Seban [4], and attributed variable property effects to a 20% increase in Nusselt number. Buoyancy (or mixed convection) effects have been considered theoretically, along with fluid property variation and blowing or suction, for the external flow of water over two-dimensional cartesian or axisymmetric bodies by Kaup and Smith [9]. Natural convection effects on forced-flow in a horizontal tube were considered, both analytically and experimentally, by Shannon and Depew [10]. They approximated the viscosity of ethylene glycol as an exponential variation with temperature. Combined forced and free convection in horizontal tubes with temperature-dependent viscosity has more recently been analyzed using an integral technique by Hong and Bergles [11]. For large values of their viscosity-variation parameter, $-(1/\mu)(\partial\mu/\partial t)\Delta t$, heat-transfer predictions are 50% above those of a constant viscosity analysis. Their results are in good agreement with their corresponding measurements in water and ethylene glycol.

Very recently, Ockendon and Ockendon [12] presented an analysis for suddenly heated or cooled channel flow of a Newtonian fluid with the viscosity either algebraically or exponentially dependent on temperature. Pearson [13] has also analyzed channel flow of high viscosity fluids when internal heat generation is very large. He presents a similarity solution for steady plane developing channel flow of a fluid whose viscosity varies exponentially with temperature. He also discusses unsteady flow, steady flow in pipes, radial disk flow, and flow in channels of varying depth.

Certain common features of formulation will be seen between the above discussion of variable property effects in forced-flow and the following discussion related to natural convection flows. For example, viscosity variation alone is the dominant variable property effect for many moderate and high Prandtl number liquids other than water, but the inclusion of additional property variations is necessary for gases and water.

The earliest known theoretical treatment of variable property effects in natural convection is the perturbation analysis of Hara [14] for air. The solution is applicable for small values of the perturbation parameter, $\varepsilon_H = (t_0 - t_\infty)/t_\infty$, see later notation. Sparrow [15] also considered natural convection with variable properties and variable wall temperature. At about the same time, Tanaev [16] investigated natural convection in a gas with variable viscosity, as did Plapp [17] for oils. A later investigation by Sparrow and Gregg [18], analyzed natural convection from an isothermal vertical surface for variable-property gases and liquid mercury. Their results indicated that the film temperature is adequate for most applications and they suggested a more accurate reference temperature for more extreme conditions. Using a successive approach method, Hara [19] extended the range of applicability of his previous perturbation solution to $\varepsilon_H = 2$ and 4.

The non-ideal-gas behavior of steam was incorporated in an analysis of natural convection from a vertical isothermal surface by Minkowycz and Sparrow [20]. They found a reference temperature coefficient of 0.46 instead of 0.38 found by Sparrow and Gregg [18]. Note that using the average (or film) reference temperature corresponds to a coefficient of 0.50.

Variable property effects in water and carbon dioxide at supercritical pressures was analyzed by Nishikawa and Ito [21], also for natural convection adjacent to a vertical isothermal surface.

Two methods of correlating the effects of variable properties on heat transfer for natural convection from vertical surfaces in liquids were examined by Fujii *et al.* [22]. They presented extensive experimental data for natural convection from a vertical cylinder with isothermal and uniform heat flux boundary conditions in laminar, transition and turbulent flow. Test data was taken in water, spindle oil and Mobiltherm 600 oil. They concluded that since the boundary-layer thickness was small compared to the radius of the cylinder, the heat-transfer coefficients should be within 1.3% of those for the corresponding flat plate problem. The first method of correlating the data consisted of using the constant property correlations for Nusselt number and evaluating all properties at a reference temperature, $t_e = t_0 - 1/4(t_0 - t_\infty)$. They noted that this choice of reference temperature agrees with that suggested by solutions of the laminar natural convection boundary-layer equations presented in two previous studies: the approximate solutions of Fujii [23] for ethylene glycol and the numerical solutions of Akagi [24] for mineral oils. The second method that Fujii *et al.* [22] used to correlate their data in oils was first proposed by Akagi [24] and applies only to liquids for which viscosity variation is dominant. This amounts to a Nusselt number correction factor which consists of the ratio of kinematic viscosities at the surface and ambient temperatures raised to the 0.21 power. The resulting correlation has the form, $Nu_x(\nu_0/\nu_\infty)^{0.21} \propto (Gr_x Pr)^{1/4}$, with all other fluid properties evaluated at the ambient fluid temperature, t_∞. For the uniform heat flux surface they transformed the correlation to incorporate a flux Grashof number and the resultant exponent of the viscosity ratio became 0.17. Excellent agreement was found between their data and both methods of correlation.

The similarity analysis of Piau [25] also treated variable property effects in natural convection from vertical surfaces in moderate and high Prandtl number liquids. He points out that the main property variations in water at moderate temperature levels are in viscosity, μ, and the volumetric coefficient of thermal expansion, β, and that for higher Prandtl number liquids, the variation of β is often negligible. The formulation considers a general variation of μ and β with temperature, but calculations are performed for a linear variation. Results are presented for water ($\sigma_f = 7.03$, 5.45 and 3.59) for two temperature differences and three temperature levels, and for the limiting situation of $\sigma_f \to \infty$. In a follow-up paper, Piau [26], includes the effect of thermal stratification of the ambient in an analysis which also includes variable μ and β effects. Corresponding measurements of the temperature field in stratified water are in good agreement with theoretical results, and an attempt is made to correlate transition to turbulence with parameters characterizing property variation.

Barrow and Sitharamarao [27] examined the effect of variable β on natural convection in water, but ignore the temperature dependence of absolute viscosity, which is known to be important. Brown [28] used an integral method with variable β and ρ but also overlooked the important variation of μ. Natural convection from a vertical, uniform-heat-flux surface was investigated experimentally and using an integral method by Ito *et al.* [29]. The temperature variation of all relevant fluid properties was included for carbon dioxide near its critical point and for spindle oil and Mobiltherm oil at atmospheric pressure. Their results for Mobiltherm oil agree well with the constant property results evaluated at the reference temperature suggested by Fujii *et al.* [22], $t_e = t_0 - 1/4(t_0 - t_\infty)$. Measurements by Booker [30] in a horizontal layer of high Prandtl number oil, experiencing a 300-fold viscosity variation, indicate only a 12% reduction in heat transfer below that of a corresponding constant viscosity fluid.

Carey and Mollendorf [31] have shown the mathematical forms of viscosity variation with temperature which result in similarity solutions for laminar natural convection from a vertical isothermal surface in liquids with temperature dependent viscosity. For the simple case of a linear variation of viscosity with temperature they presented numerical results for a range of their viscosity parameter,

$$\gamma_f \equiv \left(\frac{1}{\mu}\frac{d\mu}{dt}\right)_f (t_0 - t_\infty),$$

from -1.6 to $+1.6$ for values of film Prandtl number, σ_f, from 1 to 1000.

Considerably less work has been done concerning variable property effects on constant buoyancy natural convection flows: the plane plume above a horizontal line heat source and the flow above a horizontal line heat source on a vertical adiabatic surface. Spalding and Cruddace [32] evaluated the effect of temperature-dependent viscosity on the laminar plane plume flow above a line heat source in a fluid of large Prandtl number. They concluded that the temperature dependence of the viscosity has no influence on the flow because, for large Prandtl number, the region of non-uniform temperature is thin and concentrated in a region of small shear stress. Liburdy and Faeth [33] treated variable property effects for both the laminar plane plume and the horizontal line heat source on an adiabatic surface by assuming $\rho\mu = \rho_\infty\mu_\infty$ and $\rho k = \rho_\infty k_\infty$. Through the use of a Howarth transformation they were able to reduce the variable property problem to the equivalent constant property similarity solution equations obtained by Fujii *et al.* [34]. While the form they assumed for the property variations may be somewhat applicable to gases, it is not characteristic of most liquids.

Carey and Mollendorf [31] have shown that, if the viscosity is expanded as a Taylor series in temperature about the film temperature, a similarity solution may be obtained for vertical laminar natural convection from an isothermal surface. A necessary condition for similarity is that $t_0 - t_\infty$ be independent of x, which essentially requires that t_0 be constant in an unstratified medium. Three vertical boundary-layer flows of considerable practical importance where t_0 is variable in the downstream direction are: a uniform heat flux surface, an adiabatic surface with a concentrated energy source along the leading edge and a plane plume arising from a concentrated horizontal thermal source. Here we use a perturbation method to analyze the effect of temperature dependent viscosity on the above three vertical plane flows. The perturbation parameter is

$$\gamma_f^* = \left(\frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}t}\right)_f (t_0 - t_\infty)_0$$

where γ_f^* is evaluated at the film temperature; and the centerline temperature difference, $(t_0 - t_\infty)_0$, is that which the flow would have assuming constant viscosity evaluated at the film temperature. Since t_0 and therefore γ_f^* vary with x, these will be non-similar solutions. However, the only difference in the formulations for the three flows are the boundary conditions and the coefficients in the differential equations. The formulation, given in the next section, considers a general power-law dependence of temperature difference with downstream distance, i.e. $t_0 - t_\infty = Nx^n$. The three flow configurations correspond to particular values of n.

Computed first-order perturbation quantities are presented for these three non-similar flows for Prandtl numbers ranging from 5 to 500. In addition, first order perturbation quantities for the isothermal surface condition have been calculated for Prandtl numbers of 10 and 100. The results for the isothermal surface are compared with the similarity solution of Carey and Mollendorf [31] to determine the range of γ_f^* for which they are valid. The use of this formulation makes it possible to analyze the effect of temperature dependent viscosity on these flows in a unified manner.

FORMULATION

The present formulation assumes steady, two-dimensional (plane), vertical natural convection flow and incorporates the usual Oberbeck–Boussinesq and boundary-layer assumptions. The absolute viscosity, μ, is taken to be variable in the force–momentum balance while the fluid volumetric coefficient of thermal expansion, β, specific heat, c_p, and thermal conductivity, k, are assumed to be constant. Viscous dissipation, motion pressure and volumetric energy source effects are neglected. This results in the following governing equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1a}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) + g\beta(t - t_\infty) \tag{1b}$$

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 t}{\partial y^2} \tag{1c}$$

where u and v are the vertical and horizontal velocity components respectively and g is taken to be in the negative x direction for heated flows. The temperature of the quiescent ambient fluid, t_∞, at large values of y is taken to be constant. Except for μ, the fluid properties in (1b) and (1c) are viewed as constants to be evaluated at some reference temperature.

The absolute viscosity, μ, is expanded in a Taylor series about its value at the film temperature:

$$\mu = \mu_f + \left(\frac{\mathrm{d}\mu}{\mathrm{d}t}\right)_f (t - t_f) + \frac{1}{2}\left(\frac{\mathrm{d}^2\mu}{\mathrm{d}t^2}\right)_f (t - t_f)^2 + \ldots \tag{2}$$

This particular form is chosen to allow definition of the stream function based on the absolute viscosity at the film temperature, μ_f. For liquids, all transport properties vary with temperature. However, for many liquids, (petroleum oils, glycerin, glycols, silicone fluids, and some molten salts, for example) the percent variation of absolute viscosity with temperature is much greater than that of the other properties. Under these conditions, an analysis incorporating the above assumptions describes the momentum and thermal transport more accurately than the usual assumption of constant properties evaluated at some reference temperature. For some liquids, properties other than μ vary strongly with temperature. In particular, water and methyl alcohol exhibit strong variation of both μ and β. Furthermore, the assumptions made here are not justified for gases since the variation of other gas properties with temperature cannot be neglected.

The viscous shear term in (1b) can be expanded

$$\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} + \frac{1}{\rho}\frac{\mathrm{d}\mu}{\mathrm{d}t}\frac{\partial t}{\partial y}\frac{\partial u}{\partial y} \tag{3}$$

and, after substitution, the momentum equation becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta(t - t_\infty) + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} + \frac{1}{\rho}\frac{\mathrm{d}\mu}{\mathrm{d}t}\frac{\partial t}{\partial y}\frac{\partial u}{\partial y}. \tag{4}$$

The following generalizations are introduced to obtain the equations in terms of the generalized stream and temperature functions f and ϕ:

$$\eta(x, y) = yb(x), \quad \psi = \frac{\mu_f}{\rho}c(x)f(\eta, x) \tag{5a}$$

$$\phi(\eta, \mathrm{x}) = \frac{\mathrm{t} - \mathrm{t}_\infty}{(t_0 - t_\infty)_0}, \quad (t_0 - t_\infty)_0 = \mathrm{d}(x) = Nx^n \tag{5b}$$

$$c(x) = 4xb(x) = 4\left[\frac{g\beta\rho^2 x^3 (t_0 - t_\infty)_0}{4\mu_f^2}\right]^{1/4} \tag{5c}$$

$$= 4\left[\frac{Gr_x}{4}\right]^{1/4} \quad \gamma_f^* = \left(\frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}t}\right)_f (t_0 - t_\infty)_0 \tag{5d}$$

where $(t_0 - t_\infty)_0$ is the downstream temperature difference (along the x axis) which would result for $\gamma_f^* = 0$. Similarly, the term $(1/\mu_f)(\mathrm{d}\mu/\mathrm{d}t)_f$ is evaluated at the film temperature of the flow, $t_f = (t_0+t_\infty)_0/2$, when $\gamma_f^* = 0$.

In terms of the variables defined in (5), the expansion for viscosity may be written as

$$\mu = \mu_f[1+\gamma_f^*(x)(\phi-\tfrac{1}{2})+a_2[\gamma_f^*(x)]^2(\phi-\tfrac{1}{2})^2 + a_3[\gamma_f^*(x)]^3(\phi-\tfrac{1}{2})^3+\ldots] \quad (6a)$$

where the a_n are properties of the fluid given by

$$a_n = \frac{(\mu_f)^{n-1}}{n!}\left(\frac{d^n\mu}{dt^n}\right)_f \bigg/ \left(\frac{d\mu}{dt}\right)_f^n. \quad (6b)$$

Since $(t_0-t_\infty)_0$ must be non-zero, γ_f^* can be zero only if $(d\mu/dt)_f = 0$. For virtually all Newtonian liquids, the variation of viscosity with temperature is monotonic, i.e. no extrema arise, and $d\mu/dt$ can be zero only if μ is a constant, whereupon the higher order derivatives are also zero. Therefore, if $d\mu/dt = 0$, then $\gamma_f^* = 0$, the coefficients $a_n(\gamma_f^*)^n = 0$ and μ is a constant, μ_f. Hence the expansion (6a) is well behaved for $d\mu/dt \to 0$.

Expansions for the stream and temperature functions $f(\eta, x)$ and $\phi(n, x)$ are postulated as

$$f(\eta, \gamma_f^*) = f(\eta, x) = f_0(\eta)+\gamma_f^*(x)f_1(\eta) + [\gamma_f^*(x)]^2 f_2(\eta)+\ldots \quad (7)$$

$$\phi(\eta, \gamma_f^*) = \phi(\eta, x) = \phi_0(\eta)+\gamma_f^*(x)\phi_1(\eta) + [\gamma_f^*(x)]^2\phi_2(\eta)+\ldots. \quad (8)$$

Here we consider only first order terms and therefore the expansions for μ, f and ϕ are truncated after terms of order γ_f^*. In (6a) this amounts to a linear variation of viscosity with temperature as a first-order approximation. However, greater accuracy may be obtained for a specific fluid by retaining higher order terms in (6a), (7) and (8). If the variation of viscosity with temperature of the fluid is known, the additional parameters a_n may be determined from (6b).

Substituting (6a), (7) and (8) into (1c) and (4) with the generalizations in (5), the equations for f_0, ϕ_0, f_1 and ϕ_1 are then determined for any value of n.

$$f_0''' - (2n+2)(f_0')^2 + (n+3)f_0''f_0 + \phi_0 = 0 \quad (9a)$$

$$\phi_0'' + \sigma_f[(3+n)\phi_0'f_0 - 4nf_0'\phi_0] = 0 \quad (9b)$$

$$f_1''' - (8n+4)f_0'f_1' + (5n+3)f_0''f_1 + (n+3)f_0f_1'' + \phi_1 + f_0'''(\phi_0-1/2)+\phi_0'f_0'' = 0 \quad (10a)$$

$$\phi_1'' + \sigma_f[(3+n)\phi_1'f_0 - 8nf_0'\phi_1 - 4nf_1'\phi_0 + \phi_0'f_1(3+5n)] = 0. \quad (10b)$$

The fluid properties μ_f and γ_f^* are to be evaluated at the film temperature, $t_f = (t_0+t_\infty)_0/2$, and using the temperature difference, $(t_0-t_\infty)_0$, that the flow would have if the viscosity were constant ($\gamma_f^* = 0$). These are related to the actual film temperature, t_{fa}, and temperature difference, t_0-t_∞, as: $t_0-t_\infty = (t_0-t_\infty)_0\phi$ and $t_{fa} = t_f+\frac{1}{2}(t_0-t_\infty)_0[\phi(0)-1]$. The film Prandtl number, σ_f is defined as $\sigma_f = \mu_f c_p/k$, where μ_f is the previously defined film viscosity and c_p and k are evaluated at some chosen reference temperature, t_r. For the isothermal condition, γ_f^* will be seen to be identical to γ_f as defined by Carey and Mollendorf [31]. The corresponding Grashof number, Gr_x, is based on $(t_0-t_\infty)_0$. Gr_x is related to the actual physical local Grashof number by $Gr_x' = Gr_x\phi(0)$. The boundary conditions specified below are consistent with this formulation.

The relevant boundary conditions for the four flows to be analyzed here are as follows, where the primes indicate differentiation with respect to η.

(a) Isothermal surface with horizontal leading edge, $n = 0$

$$\phi(\infty, x) = f'(0, x) = f(0, x) = f'(\infty, x) = 1-\phi(0, x) = 0 \quad (11a)$$

$$1-\phi_0(0) = \phi_0(\infty) = f_0'(0) = f_0(0) = f_0'(\infty) = 0 \quad (11b)$$

$$\phi_1(0) = \phi_1(\infty) = f_1'(0) = f_1(0) = f_1'(\infty) = 0. \quad (11c)$$

(b) Uniform-flux surface with a horizontal leading edge, $n = 1/5$

$$\phi(\infty, x) = f'(0, x) = f(0, x) = f'(\infty, x) = 0 \quad (12a)$$

$$1-\phi_0(0) = \phi_0(\infty) = f_0'(0) = f_0(0) = f_0'(\infty) = 0 \quad (12b)$$

$$\phi_1'(0) = \phi_1(\infty) = f_1'(0) = f_1(0) = f_1'(\infty) = 0. \quad (12c)$$

(c) An adiabatic surface with a concentrated heat source along the horizontal leading edge, $n = -3/5$.

$$\phi'(0, x) = f'(0, x) = f(0, x) = f'(\infty, x) = 0 \quad (13a)$$

$$1-\phi_0(0) = \phi_0'(0) = f_0'(0) = f_0(0) = f_0'(\infty) = 0 \quad (13b)$$

$$\phi_1'(0) = \phi_1(\infty) = f_1'(0) = f_1(0) = f_1'(\infty) = 0. \quad (13c)$$

(d) A plane plume rising from a horizontal thermal source, $n = -3/5$.

$$\phi'(0, x) = f(0, x) = f''(0, x) = f'(\infty, x) = 0 \quad (14a)$$

$$1-\phi_0(0) = \phi_0'(0) = f_0(0) = f_0''(0) = f_0'(\infty) = 0 \quad (14b)$$

$$\phi_1'(0) = \phi_1(\infty) = f_1(0) = f_1''(0) = f_1'(\infty) = 0. \quad (14c)$$

For the isothermal condition, $n = 0$, and since $\phi_1(0) = 0$, the temperature at $y = 0$ is not altered by varying γ_f^*. Consequently, the film temperature, $t_f = (t_0 + t_\infty)/2$, and $t_0 - t_\infty$ are not altered by varying γ_f^*. Therefore, for the isothermal condition, γ_f^* is equal to $\gamma_f = (1/\mu)_f(\mathrm{d}\mu/\mathrm{d}t)_f(t_0 - t_\infty)$ as defined by Carey and Mollendorf [31]. The values of n shown above for the other three flow conditions are determined by calculating the value of $Q(x)$, the total heat convected in the flow at any downstream location x.

$$Q(x) = \int_0^\infty \rho c_p(t - t_\infty)u\,\mathrm{d}y$$
$$= \mu_f c_p c d \int_0^\infty \phi f'\,\mathrm{d}\eta \propto x^{(3+5n)/4}. \quad (15)$$

This must increase linearly with x for the uniform heat flux surface condition, (b), and be independent of x for the adiabatic flows, (c) and (d). Therefore,

$$n_a = 0 \quad (16a)$$
$$n_b = 1/5 \quad (16b)$$
$$n_c = n_d = -3/5. \quad (16c)$$

Including the first order terms in f and ϕ for $\gamma_f^* \neq 0$, $Q(x)$ is

$$Q(x) = \mu_f c_p c d\left[\int_0^\infty \phi_0 f_0'\,\mathrm{d}\eta + \gamma_f^* \int_0^\infty (\phi_0 f_1' + \phi f_0')\,\mathrm{d}\eta\right]. \quad (17)$$

For the uniform flux condition, (b), and the adiabatic flows, (c) and (d), integration of the first order energy equation (10b) shows that the second integral in (17) is zero. This is required to ensure that additional x dependence is not added to $Q(x)$ through γ_f^*. $Q(x)$ may therefore be written as

$$Q(x) = \mu_f c_p c d I_Q \text{ where } I_Q = \int_0^\infty \phi_0 f_0'\,\mathrm{d}\eta. \quad (18)$$

The mass flow per unit width of surface, $\dot{m}$, becomes

$$\dot{m} = \int_0^\infty \rho u\,\mathrm{d}y$$
$$= \mu_f c \int_0^\infty f'\,\mathrm{d}\eta = \mu_f c[f_0(\infty) + \gamma_f^* f_1(\infty)] \quad (19)$$

and the momentum flux in the x direction is given by:

$$M(x) = \int_0^\infty \rho u^2\,\mathrm{d}y = \frac{\mu_f^2 c^2 b}{\rho}\int_0^\infty (f')^2\,\mathrm{d}y$$
$$= \frac{\mu_f^2 c^2 b}{\rho}[I_{M0} + \gamma_f^* I_{M1}] \quad (20)$$

where

$$I_{M0} = \int_0^\infty (f_0')^2\,\mathrm{d}\eta \text{ and } I_{M1} = \int_0^\infty 2f_0' f_1'\,\mathrm{d}\eta.$$

For the isothermal surface γ_f^* is a constant so that the expansions (7) and (8) are simple parameter expansions. However, for the other flows considered here γ_f^* is a function of x and therefore (7) and (8) are actually coordinate expansions. For the uniform flux surface γ_f^* is proportional to $x^{1/5}$ so that the expansion is valid for small x, with the effect of variable viscosity increasing with downstream distance. For the wall plume and free plume γ_f^* is proportional to $x^{-3/5}$ and therefore expansions (7) and (8) are valid for large x with the effect of variable viscosity increasing as $x \to 0$.

Stewartson [35] discusses a fundamental difficulty which arises when trying to obtain asymptotic solutions of the boundary-layer equations valid for large x. He points out that the parabolic nature of the boundary-layer equations leads to an arbitrariness being introduced into the asymptotic expansion at some stage as a consequence of neglecting the boundary conditions at the leading edge. He shows that to resolve this and obtain a solution which is exponentially small as $\eta \to \infty$ it is often necessary to include logarithmic terms in the asymptotic expansion. Consequently, when developing higher order terms in the expansions of $f(\eta, \gamma^*)$ and $\phi(\eta, \gamma^*)$ for the plume flows, one must consider the possibility of logarithmic terms in γ_f^* as well as powers of γ_f^*. However, for the isothermal and uniform-flux surfaces we expect higher order terms to be only increasing powers of γ_f^*.

CALCULATIONS

The perturbation analysis provides equations valid for $|\gamma_f^*|$ small. Comparison of these results for the isothermal conditions with the more exact similarity solution of Carey and Mollendorf [31] is an indication of the accuracy, or range of validity of the present perturbation analysis.

Previous works that have dealt with the constant property analysis of the four flows considered here, have employed different mathematical formulations. The formulation (9), with the appropriate boundary conditions and value of n, corresponds to the constant-property analysis of Gebhart [36] for the isothermal surface condition [(11b) and (16a)], the uniform heat-flux condition [(12b) and (16b)] and the plane plume above a horizontal source [(14b) and (16c)]. The equations in (9) with (13b) and (16c) correspond to the constant property analysis of Jaluria and Gebhart [37] for the flow above a horizontal line source on an adiabatic vertical surface.

Numerical integration of equations (9) and (10) with the appropriate boundary conditions was done using a predictor-corrector to integrate and the technique described by Nachtsheim and Swigert [38] to correct the initial guesses. The numerical integration scheme employed automatic local subdivision of the independent variable, η, to ensure prescribed accuracy. An accuracy criterion of 10^{-10} was used and η_{edge} was increased to as large as 70 so that all results were unchanging to five digits. For the non-similar flows, calculations were carried out for values of Prandtl number of 5, 10, 50, 100 and 500 which are representative of many common liquids with temperature-dependent viscosity. For the isothermal boundary condition, the present perturbation solution results are given for Prandtl numbers of 10 and 100.

RESULTS

For the first order, linear variation of viscosity with temperature it can be shown that

$$\gamma_f^* = \left(\frac{\mu_0}{\mu_\infty} - 1\right) \Big/ \left[\phi(0) + 1/2\left(\frac{\mu_0}{\mu_\infty} - 1\right)\right] \quad (21)$$

where μ_0 and μ_∞ are the viscosity at t_0 and t_∞, respectively. The relationship between γ_f^* and μ_0/μ_∞ is not explicit since $\phi(0)$ is implicitly a function of γ_f^*.

Since for most liquids β is greater than zero and $(1/\mu)(\mathrm{d}\mu/\mathrm{d}t)$ is less than zero, $\gamma_f^* < 0$ usually corresponds to $t_0 > t_\infty$ (heated flows) and upward flow with $\mu_0 < \mu_\infty$. The most common case for $\gamma_f^* > 0$ is $t_0 < t_\infty$ (cooled flows) and downward flow with $\mu_0 > \mu_\infty$.

The stream function, as defined here, is based on the film viscosity. For the flows adjacent to a vertical surface, the shear stress at the surface, $\tau_0(x)$, is therefore a function of γ_f^* directly, as well as through $f''(0)$.

$$\tau_0(x) = \mu_f^2(4/x^2)(Gr'_x/4)^{3/4}\tau^*/\rho \quad (22)$$

where $\tau^* = [1+\phi(0)\gamma_f^*/2]f''(0)/[\phi(0)]^{3/4}$. Substituting the expansions for ϕ and f'' and keeping only first order terms in γ_f^* yields

$$\tau^* = [(1+\gamma_f^*/2)f_0''(0) + \gamma_f^* f_1''(0)]/[\phi_0(0)+\gamma_f^*\phi_1(0)]^{3/4}. \quad (23)$$

The numerical results of the perturbation analysis for the four flow configurations corresponding to the indicated values of Prandtl number are summarized in Table 1. For the isothermal condition, Table 2 provides a detailed comparison of the perturbation results with those for the corresponding similarity solution. It can be seen that for $|\gamma_f^*|$ as large as 0.8 the perturbation results for the heat transfer, $\phi'(0)$; mass flow, $f(\infty)$; and wall shear τ^*, parameters agree within 2% of the similarity solution results. These results imply that despite the initial assumptions of small $|\gamma_f^*|$, the perturbation analysis provides meaningful results for $|\gamma_f^*| \leqslant 0.8$. Figure 1 shows f_0', ϕ_0, f_1' and ϕ_1, for the isothermal surface condition with $\sigma_f = 100$. Also shown is a comparison of the velocity and temperature profiles predicted by the similarity solution and the perturbation analysis for $\sigma_f = 100$ and $\gamma_f^* = -0.8$. The respective profiles of the two analyses are seen to agree well across the flow field.

The effect of non-zero γ_f^* on the velocity and temperature profiles for the uniform heat flux surface can be seen in Fig. 2 for $\gamma_f^* = -0.8$, 0, 0.8 and $\sigma_f = 100$. The profiles for $\gamma_f^* = 0$ correspond to f_0' and ϕ_0. ϕ_1 and f_1' are also shown. The trends for this flow are seen to be similar to the isothermal surface condition.

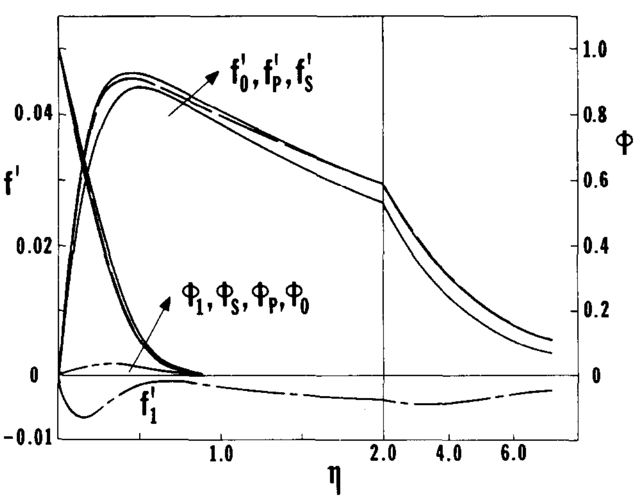

FIG. 1. For the isothermal surface condition with $\sigma_f = 100$, ϕ_0, ϕ_1, f_0' and f_1' and a comparison of the perturbation solution (ϕ_p and f_p') with the similarity solution (ϕ_s and f_s') for $\gamma_f^* = -0.8$.

Table 1. For the perturbation solutions, calculated flow and transport quantities for various n and σ_f as indicated

Isothermal surface $n = 0$

σ_f	$f_0''(0)$	$f_1''(0)$	$\phi_0'(0)$	$\phi_1'(0)$	$f_0(\infty)$	$f_1(\infty)$
10	0.41920	−0.16465	−1.16933	0.07625	0.24923	−0.04885
100	0.25169	−0.10438	−2.19137	0.16279	0.13664	−0.03313

Uniform flux surface $n = 1/5$

σ_f	$f_0''(0)$	$f_1''(0)$	$\phi_0(0)$	$\phi_1(0)$	$f_0(\infty)$	$f_1(\infty)$
5	0.45443	−0.15933	−1.07630	0.04934	0.27644	−0.04037
10	0.39503	−0.14138	−1.31642	0.05216	0.22719	−0.03823
50	0.27779	−0.10291	−2.04739	0.05745	0.14860	−0.02943
100	0.23668	−0.08855	−2.45844	0.05909	0.12448	−0.02554
500	0.16121	−0.06121	−3.72496	0.06161	0.08289	−0.01782

Line source plume $n = -3/5$

σ_f	$f_0'(0)$	$f_1'(0)$	$\phi_1(0)$	$f_0(\infty)$	$f_1(\infty)$	I_Q	I_{M0}	I_{M1}
5	0.47393	−0.00365	−0.01457	0.56380	−0.09529	0.22566	0.17034	−0.00900
10	0.41395	0.00269	−0.02400	0.50156	−0.11013	0.15387	0.12735	−0.01136
50	0.29352	0.01010	−0.04440	0.39762	−0.11868	0.06026	0.06667	−0.01135
100	0.25066	0.01083	−0.05079	0.36253	−0.11545	0.03972	0.05085	−0.01001
500	0.17131	0.00970	−0.06039	0.29449	−0.10172	0.01485	0.02740	−0.00652

Line source on adiabatic surface $n = -3/5$

σ_f	$f_0''(0)$	$f_1''(0)$	$\phi_1(0)$	$f_0(\infty)$	$f_1(\infty)$	I_Q	I_{M0}	I_{M1}
5	0.69236	−0.18996	0.11350	0.46367	−0.12865	0.16901	0.078160	−0.023711
10	0.61060	−0.16864	0.12636	0.38357	−0.13415	0.10676	0.048987	−0.018718
50	0.43947	−0.12445	0.14750	0.25304	−0.11956	0.03465	0.015907	−0.008510
100	0.37679	−0.10771	0.15378	0.21241	−0.10767	0.021005	0.009668	−0.005625
500	0.25895	−0.07516	0.16330	0.14184	−0.07894	0.006448	0.002982	−0.001945

Table 2. Comparison of present analysis, p, with the similarity solution of Carey and Mollendorf [31], s, for an isothermal vertical surface and small γ_f^*

σ_f	γ_f^*	$\dfrac{f_s''(0)}{f_p''(0)}$	% Error	$\dfrac{\phi_s'(0)}{\phi_p'(0)}$	% Error	$\dfrac{f_s(\infty)}{f_p(\infty)}$	% Error	$\dfrac{\tau_s^*}{\tau_p^*}$	% Error
10	−0.8	0.62655	13.7	−1.24762	1.4	0.28834	0.0	0.37593	1.9
		0.55092		−1.23033		0.28830		0.38324	
	0.8	0.32190	10.7	−1.11899	1.0	0.20829	0.9	0.45066	1.0
		0.28748		−1.10833		0.21014		0.45516	
100	−0.8	0.38477	13.9	−2.36000	1.6	0.16324	0.1	0.23086	1.6
		0.33519		−2.32160		0.16314		0.23450	
	0.8	0.19039	11.7	−2.08434	1.1	0.10851	1.5	0.26655	0.9
		0.16819		−2.06114		0.11014		0.26887	

The effect of γ_f^* on heat transfer for the isothermal and uniform flux surfaces is next considered. A heat-transfer parameter is related to $\phi'(0, \gamma_f^*)$ as follows; and the surface heat flux, q'', and local Nusselt number, Nu_x, are determined as

$$q'' = -k \left.\frac{\partial t}{\partial y}\right|_{y=0}$$

$$= [-\phi'(0, \gamma_f^*)](t_0 - t_\infty)_0 \frac{k}{4x}\left(\frac{Gr_x}{4}\right)^{1/4} \tag{24}$$

$$\frac{q''}{(t_0 - t_\infty)}\left(\frac{x}{k}\right) = Nu_x = \left[\frac{-\phi'(0, \gamma_f^*)}{\sqrt{2}}\right]\frac{(Gr_x')^{1/4}}{[\phi(0, \gamma_f^*)]^{5/4}}. \tag{25}$$

The above relation is rewritten for convenience as

$$N' = \frac{\sqrt{2}Nu_x}{(Gr_x')^{1/4}} = [-\phi'(0, \gamma_f^*)]/[\phi(0, \gamma_f^*)]^{5/4}. \tag{26}$$

Figure 3 shows the values of the heat-transfer parameter, N', predicted by the perturbation analysis

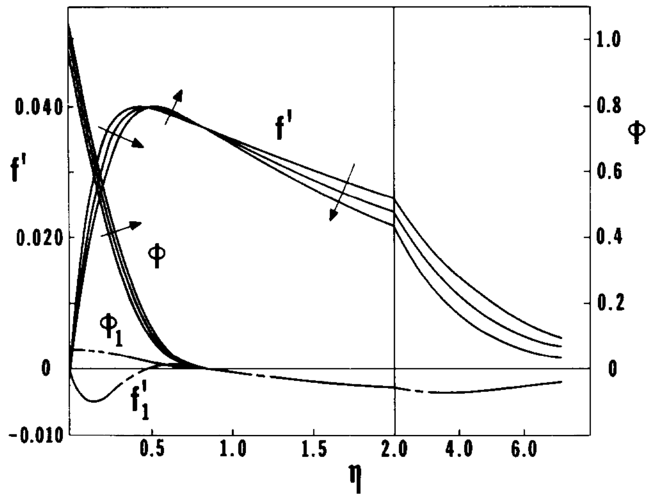

FIG. 2. For the uniform heat flux surface condition with $\sigma_f = 100$, the effect of γ_f^* on $\phi(\eta, \gamma_f^*)$ and $f'(\eta, \gamma_f^*)$ is shown for $\gamma_f^* = -0.8, 0, 0.8$. Curves for $\gamma_f^* = 0$ correspond to f_0' and ϕ_0. Also shown are f_1' (— — —) and ϕ_1 (– – – – –). The arrows indicate the direction of increasing γ_f^*.

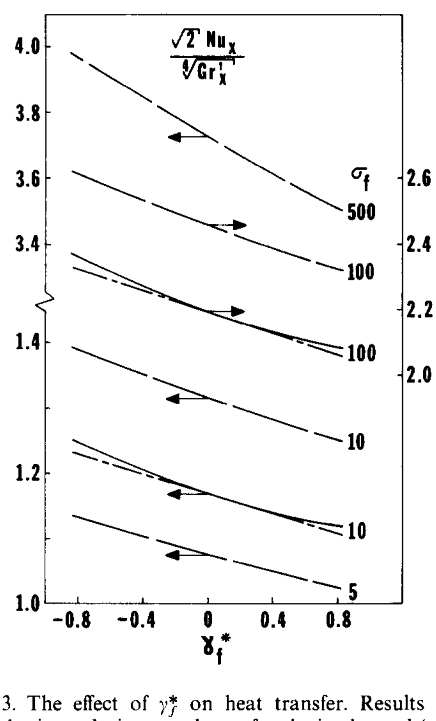

FIG. 3. The effect of γ_f^* on heat transfer. Results of the perturbation solution are shown for the isothermal (— — —) and uniform heat flux (— —) surface conditions for the indicated values of film Prandtl number. Also shown for the isothermal condition are the results of the similarity solution (—).

for the isothermal and uniform heat flux surfaces. Also shown for the isothermal surface are the N' results for the similarity solutions. Good agreement between the perturbation and similarity solution results is seen for the range of γ_f^* considered here, i.e. $|\gamma_f^*| \leqslant 0.8$. For both the isothermal and uniform heat flux surfaces, $\gamma_f^* < 0$ increases the surface heat transfer while $\gamma_f^* > 0$ reduces it. Even for the limited range of $|\gamma_f^*| \leqslant 0.8$ the effects on transport for the isothermal and uniform heat flux surfaces are significant. For $|\gamma_f^*| = 0.8$, the deviation from the constant viscosity ($\gamma_f^* = 0$) result is as much as 7% for the heat-transfer parameter, N'; 10% for surface shear, τ^*; and 15% for mass flow, $f(\infty)$.

The heat-transfer correlations of Fujii *et al.* [22] for laminar natural convection from a vertical flat plate in moderate and high Prandtl number liquids with temperature dependent viscosity are

$$(Nu_x)_\infty (\nu_0/\nu_\infty)^{0.21} = 0.49(Gr_x\sigma)_\infty^{1/4} \qquad (27)$$

for the isothermal surface and

$$(Nu_x)_\infty (\nu_0/\nu_\infty)^{0.17} = 0.62(Gr_x^*\sigma)_\infty^{1/4} \qquad (28)$$

for the uniform heat flux surface, where $Gr_x^* = g\beta q''x^4/k\nu^2$ is the flux Grashof number in which fluid properties are evaluated at t_∞. Using the assumption of linear dependence of viscosity on temperature and assuming $\mu_\infty/\mu_0 = \nu_\infty/\nu_0$, the results of the present analysis can be converted to a form that can be compared to these correlations. For this comparison, the reference temperature for properties other than μ in the present analysis are evaluated at t_∞. With these assumptions, one can determine the specific values of ν_∞/ν_0 and σ_∞ which correspond to particular values of γ_f^* and σ_f. These are determined as follows

$$\nu_\infty/\nu_0 = \mu_\infty/\mu_0 = (1-\gamma_f^*/2)/\{1+\gamma_f^*[\phi(0)-1/2]\} \qquad (29)$$

$$\sigma_\infty = \sigma_f(1-\gamma_f^*/2) \qquad (30)$$

where $\phi(0) = 1+\gamma_f^*\phi_1(0)$. For the isothermal surface, the correlation of Fujii *et al.* [22] may be written as

$$\begin{aligned} H_F &= (Nu_x)_\infty/(Gr_x)_\infty^{1/4} \\ &= 0.49(\nu_\infty/\nu_0)^{0.21}\sigma_\infty^{1/4} \end{aligned} \qquad (31)$$

Table 3. A comparison of the local heat-transfer parameters predicted by the present analysis and the similarity solution of Carey and Mollendorf [31] CM, with those predicted by the correlations of Fujii *et al.* [22] F, for the isothermal and uniform heat flux conditions

(a) Isothermal

σ_f	γ_f^*	ν_∞/ν_0	σ_∞	H_{CM}	H_F	% Difference
100	−0.8	2.333	140	1.942	2.014	3.5
	−0.4	1.500	120	1.748	1.766	1.0
	0	1.000	100	1.550	1.550	0.0
	0.4	0.667	80	1.345	1.346	0.0
	0.8	0.429	60	1.129	1.141	1.1
100†	−1.6	9.000	180	2.577	2.847	9.5
	−0.8	2.333	140	1.975	2.014	1.9
	0.8	0.429	60	1.142	1.141	0.0
	1.6	0.111	20	0.636	0.653	2.6
1000†	−1.6	9.000	1800	4.727	5.063	6.6
	−0.8	2.333	1400	3.588	3.581	0.2
	0	1.000	1000	2.804	2.755	1.8
	0.8	0.429	600	2.060	2.030	1.5
	1.6	0.111	200	1.144	1.162	1.5

(b) Uniform heat flux

σ_f	γ_f^*	ν_∞/ν_0	σ_∞	H_{CM}^*	H_F^*	% Difference
50	−0.8	2.199	70	1.612	1.658	2.8
	−0.4	1.483	60	1.480	1.504	1.6
	0	1.000	50	1.344	1.356	0.8
	0.4	0.662	40	1.202	1.209	0.5
	0.8	0.418	30	1.048	1.055	0.7
100	−0.8	2.195	140	1.869	1.904	1.8
	−0.4	1.482	120	1.715	1.727	0.7
	0	1.000	100	1.556	1.557	0.1
	0.4	0.661	80	1.391	1.388	0.2
	0.8	0.417	60	1.211	1.212	0.1
500	−0.8	2.189	700	2.611	2.626	0.5
	−0.4	1.482	600	2.393	2.383	0.4
	0	1.000	500	2.170	2.149	1.0
	0.4	0.661	400	1.937	1.915	1.1
	0.8	0.417	300	1.686	1.672	0.8

†Similarity solutions results of Carey and Mollendorf [31].

which may be compared to the transformed expression of the present analysis for the isothermal surface

$$\begin{aligned} H_{CM} &= (Nu_x)_\infty/(Gr_x)_\infty^{1/4} \\ &= [-\phi'(0)](1-\gamma_f^*/2)^{1/2}/\sqrt{2} \end{aligned} \quad (32)$$

The corresponding relations for the uniform heat flux surface are

$$H_F^* = (Nu_x)_\infty/(Gr_x^*)_\infty^{1/5} = 0.62(\nu_\infty/\nu_0)^{0.17}\sigma_\infty^{1/5} \quad (33)$$

and

$$\begin{aligned} H_{CM}^* &= (Nu_x)_\infty/(Gr_x^*)_\infty^{1/5} \\ &= [-\phi'(0)]^{4/5}[1-\gamma_f^*/2]^{2/5}/\{(2)^{2/5}[\phi(0)]\}. \end{aligned} \quad (34)$$

When comparing correlations, $-\phi'(0)$ and $\phi(0)$ are determined at the values of σ_f and γ_f^* which correspond to ν_∞/ν_0 and σ_∞, using (29) and (30). Table 3 shows a comparison of H_F with H_{CM} and H_F^* with H_{CM}^* for the indicated values of γ_f^* and σ_f. The results of the perturbation analysis are seen to agree extremely well with the experimental correlations of Fujii *et al.* [22] for both the isothermal and uniform heat flux surface conditions. Also shown in Table 3 for the isothermal surface are the similarity solution results of Carey and Mollendorf [31] for $\sigma_f = 100$ and 1000. The similarity solution results are in slightly better agreement with those of Fujii *et al.* [22] than those of the perturbation analysis for $|\gamma_f^*| \leq 0.8$. Even for values of $|\gamma_f^*|$ as large as 1.6 there is good agreement between the similarity solution and the correlation of Fujii *et al.* [22]. It is interesting to note that for $|\gamma_f^*| = 0.8$ the percent change, compared to the constant viscosity results, ($\gamma_f^* = 0$), is as much as 25% for H_{CM} and H_{CM}^*; while for N', the deviation is limited to no more than 7%. This difference occurs because, for H_{CM} and H_{CM}^*, the reference viscosity is evaluated at t_∞ while for N' it is evaluated at t_f. This demonstrates how the choice of

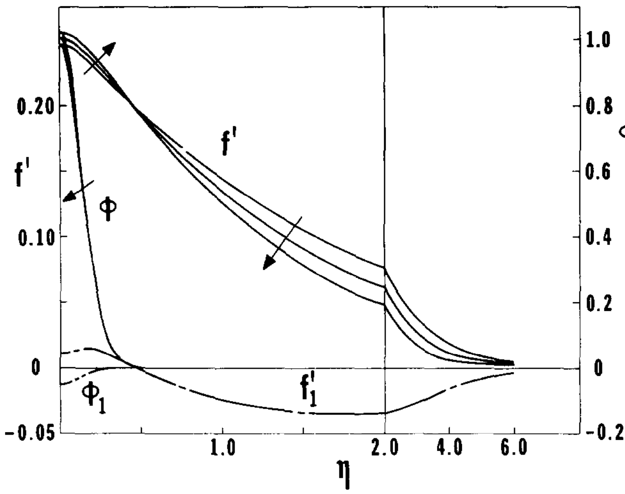

FIG. 4. For the plane plume with $\sigma_f = 100$, the effect of γ_f^* on $\phi(\eta, \gamma_f^*)$ and $f'(\eta, \gamma_f^*)$ is shown for $\gamma_f^* = -0.4$, 0, 0.4. Curves for $\gamma_f^* = 0$ correspond to f_0' and ϕ_0. Also shown are f_1' (———) and ϕ_1 (– – – –). The arrows indicate the direction of increasing γ_f^*.

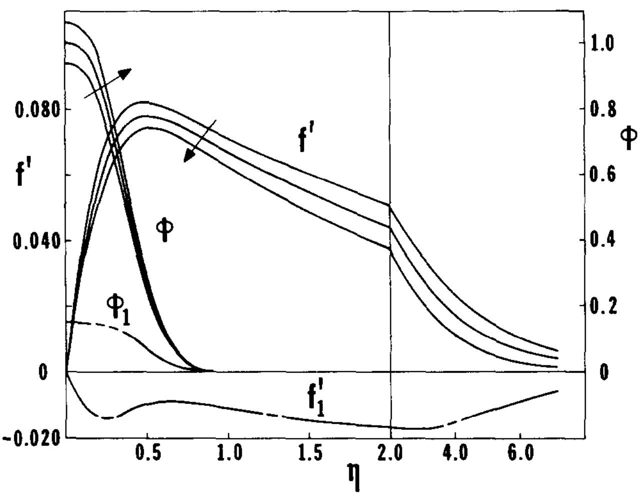

FIG. 5. For the concentrated horizontal source on an adiabatic surface with $\sigma_f = 100$, the effect of γ_f^* on $\phi(\eta, \gamma_f^*)$ and $f'(\eta, \gamma_f^*)$ is shown for $\gamma_f^* = -0.4$, 0, 0.4. Also shown are f_1' (———) and ϕ_1 (– – – –). The arrows indicate the direction of increasing γ_f^*.

reference temperature influences the predicted effect of variable viscosity on heat transfer. This forms the basis for the alternate method of correlating variable property effects on heat transfer used by Fujii *et al.* [22] in which all fluid properties are evaluated at $t_e = t_0 - 1/4(t_0 - t_\infty)$.

Figures 4 and 5 show the effect of $\gamma_f^* \neq 0$ on the temperature and velocity profiles for the flow above a horizontal line thermal source and the flow above a horizontal line thermal source on a vertical adiabatic surface, respectively. Comparison of the similarity solution to the perturbation analysis for the isothermal surface indicates that the results are in close agreement for $|\gamma_f^*|$ as large as 0.8. However, for the two adiabatic flows, the values of f_1' in the outer portion of the velocity boundary layer are so large that $\gamma_f^* > 0.4$ would predict local flow reversal [i.e. $f'(\eta, x) = f_0'(\eta) + \gamma_f^* f_1'(\eta) < 0$]. Since this calls into question the validity of the boundary-layer approximations, the range of validity of the present first order perturbation results for the two adiabatic flows is thought to be limited to $|\gamma_f^*| \leqslant 0.4$. In Figs. 4 and 5, f' and ϕ profiles are shown for $\sigma_f = 100$ and $\gamma_f^* = -0.4$, 0 and 0.4. The profiles for $\gamma_f^* = 0$ correspond to f_0' and ϕ_0, and f_1' and ϕ_1 are also shown.

It can be seen in Fig. 4 and Table 1 that for the plane plume, non-zero values of γ_f^* produce significant changes in the velocity and temperature at the centerline of the plume, even for values of Prandtl number as large as 100 and 500. For $|\gamma_f^*| = 0.4$, there is about 2% change in $f'(0)$ and $\phi(0)$ compared to the constant viscosity results, while the change in mass flow, $f(\infty)$, is as much as 14%. As previously mentioned, Spalding and Cruddace [32] concluded that the temperature dependence of viscosity has no influence on large Prandtl number flows because for large Prandtl number the region of non-uniform temperature is thin and concentrated in a region of small shear. While their conclusion seems intuitively correct, the present analysis indicates that it is not applicable for $\sigma_f \leqslant 500$. Further, it can be seen in the profiles in Fig. 4 for $\sigma_f = 100$ that the effect of temperature-dependent viscosity is not confined to the thermal boundary layer region. The greatest alteration of the velocity profile occurs in the center of the velocity boundary layer, well out beyond the thermal boundary layer. Increasing γ_f^* produces a decrease in centerline temperature of the plume which is opposite to the trend observed for the flow above a horizontal line source on a vertical adiabatic surface. In this regard it is interesting to note the centerline temperature measurements for plane plumes reported by Fujii, Morioka and Uehara [34]. Their Fig. 9 presents a plot of non-dimensional centerline temperature vs local flux Grashof number and shows their own data for air, water and spindle oil as well as the data for air of Brodowicz and Kierkus [39] and Forstrom and Sparrow [40] and the data for water and silicone fluid of Schorr and Gebhart [41]. The data of Forstrom and Sparrow [40] and Schorr and Gebhart [41] as well as the air data of Fujii *et al.* [34] all fall about 15% lower than the line predicted by the similarity solution. Lyakhov [42] has shown that this systematic deviation is due to inaccurate treatment of the entrainment near the line thermal source. The centerline temperature data of Fujii *et al.* [34] for spindle oil and water, however, are on the line predicted by the similarity solution for local flux Grashof numbers around 10^3 and the data decrease with increasing Grashof number until they are 15% low at 10^6. From the results of Lyakhov [42] all data should be about 15% low on this plot. Further, the data of Fujii *et al.* [34] for spindle oil and water were taken at values of heat flux that are an order of magnitude larger than those of the other authors mentioned here. The high values of heat flux will produce high centerline temperature differences, particularly near the source (low values of local flux Grashof number). Since both water and spindle oil have temperature dependent viscosity, the data points at smaller values of local flux Grashof number would correspond to large negative values of γ_f^*. From Table 1 it can be seen that negative values of γ_f^* produce an increase in the centerline temperature of the plane plume. This suggests that the data points for spindle oil and water are high at low values of flux Grashof number because of the effect of temperature dependent viscosity. For water, the effect of variable β must also be considered and therefore the viscosity variation is only one possible cause of the deviation. However, for oils the viscosity variation is dominant and the results of our analysis appear to explain the trend seen in the spindle oil data of Fujii *et al.* [34].

Finally, Fig. 5 shows the effect of temperature dependent viscosity on the temperature and velocity profiles for the flow above a horizontal line source on a vertical adiabatic surface with $\sigma_f = 100$. It can be seen that the effect of the viscosity variation is not limited to the thermal boundary region, but affects the velocity profile well out beyond the thermal region. Compared to the constant property results, for $|\gamma_f^*| = 0.4$, the respective differences in $\phi(0)$, τ^* and $f(\infty)$ are as much as 6%, 5% and 22%. The effect of γ_f^* and σ_f on the centerline temperature for the two adiabatic flows can be seen in Fig. 6. Increasing γ_f^* produces decreasing $\phi(0, \gamma_f^*)$ for the plane plume while the opposite trend is observed for the plane wall plume. Also shown in Fig. 6 is the effect of γ_f^* and σ_f on the centerline velocity of the plane plume. For $\sigma_f = 5$, increasing γ_f^* actually produces a slight decrease in $f'(0, \gamma_f^*)$, while for $\sigma_f \geqslant 10$, increasing γ_f^* produces a slight increase in $f'(0, \gamma_f^*)$.

CONCLUSIONS

For many liquids the variation of viscosity with temperature is much greater than the variation of other fluid properties. When this is true, the analysis presented here provides a more accurate picture of the thermal and momentum transport in the four flows considered than the usual analysis with constant properties. The truncated expansion for μ amounts to a linear variation of viscosity with temperature which is not exact for many fluids but is

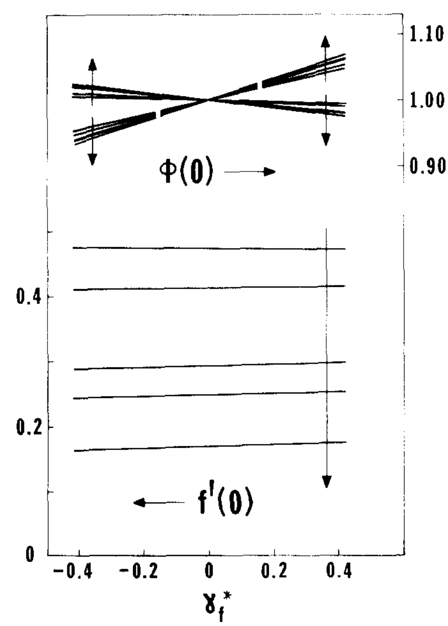

FIG. 6. The effect of Prandtl number on transport, $\phi(0, \gamma_f^*)$ for the adiabatic flows as a function of γ_f^* and on $f'(0, \gamma_f^*)$ (lower curves) for the plane plume. For the upper curves (—— plane plume and — — concentrated horizontal source on an adiabatic surface) and for the lower $f'(0, \gamma_f^*)$ curves, the arrows indicate increasing σ_f for $\sigma_f = 5$, 10, 50, 100, 500.

a good approximation for the small values of γ_f^* required for the present perturbation analysis.

Even for moderate values of γ_f^*, significant effects on thermal and momentum transport are found for all four flows considered here. The excellent agreement between the perturbation analysis results and the heat-transfer data and correlations of Fujii *et al.* [22] for the isothermal and uniform heat flux surface conditions lends strong support to the calculated results. It further suggests that for these flows, the assumption of linear variation of viscosity with temperature is adequate to predict transport for many circumstances. The results of Fujii *et al.* [22] also agree well with the similarity solution of Carey and Mollendorf [31].

In light of the conclusions of Lyakhov [42], the computed results for the plane plume seem to explain the trends in the centerline temperature data of Fujii *et al.* [34] for spindle oil at high values of source heat flux. While the present first-order analysis is only applicable to a limited range of γ_f^*, it provides considerable information about the effect of temperature dependent viscosity on these laminar natural convection flows. If needed, greater accuracy may be obtained with the present perturbation scheme by including higher-order terms in the expansions for viscosity, stream function and temperature excess ratio.

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EFFETS DE LA VISCOSITE VARIABLE SUR LES ECOULEMENTS DE CONVECTION NATURELLE

Résumé—Une analyse régulière de perturbation est présentée pour trois types de convection naturelle dans des liquides à viscosité variable avec la température: un panache plan ascensionnel, l'écoulement au dessus d'une source linéaire horizontale sur une surface adiabatique et l'écoulement adjacent à une surface verticale chauffée à flux uniforme. Alors que ces écoulements ont des solutions de similarité en loi puissance lorsque la viscosité est constante, cela n'est pas lorsque la viscosité est fonction de la température. Un écoulement à similarité, celui adjacent à une surface verticale isotherme, est analysé pour comparaison de façon à estimer le domaine de validité de l'analyse de perturbation. La formulation utilisée ici fournit un traitement unifié des effets de la viscosité variable sur ces quatre écoulements. A l'exception de l'eau, la variation principale des propriétés du fluide est relative à la viscosité absolue. Cela est connu et déjà utilisé pour d'autres écoulements, cela constitue la base de la présente étude. On présente pour les quatre cas les calculs des perturbations au premier ordre. Plusieurs comportements intéressants sont suggérés pour l'écoulement et le transfert. Ces modifications par rapport au cas de la viscosité constante sont significatives même dans les limites de la perturbation de premier ordre. Les résultats pour le transfert thermique avec des surfaces à température ou à flux constants sont en bon accord avec les données et les formules antérieures. Les résultats présentés éclairent quelques conclusions déjà énoncées sur les écoulements de panache.

EINFLÜSSE VERÄNDERLICHER VISKOSITÄT BEI EINIGEN FREIEN KONVEKTIONSSTRÖMUNGEN

Zusammenfassung—Eine analytische Lösung mit normalem Störungsansatz wird für drei laminare freie Konvektionsströmungen in Flüssigkeiten mit temperaturabhängiger Viskosität beschrieben: eine frei aufsteigende, ebene Strömung, die Strömung oberhalb einer horizontalen Linienquelle auf einer adiabaten Oberfläche (Strömung längs einer ebenen Wand) und die Strömung nahe einer vertikalen Oberfläche mit gleichförmigem Wärmestrom. Während diese Strömungen wohlbekannte Ähnlichkeitslösungen nach Potenzgesetzen haben, wenn die Viskosität des Fluids als konstant angenommen wird, sind diese nicht-ähnlich, wenn die Viskosität als Funktion der Temperatur betrachtet wird. Eine einzelne ähnliche Strömung, die nahe einer vertikalen isothermen Oberfläche ist, wird zum Vergleich ebenfalls untersucht, um den Gültigkeitsbereich der Lösung durch Störungsansatz abzuschätzen. Die hier benutzten Formeln erlauben eine gemeinsame Behandlung des Einflusses veränderlicher Viskosität bei diesen vier Strömungsformen. Von Wasser abgesehen, ist bei üblichen Flüssigkeiten die Stoffeigenschaft mit der größten Temperaturabhängigkeit die dynamische Viskosität. Dies wurde schon früher erkannt und bei der Behandlung anderer Strömungen genutzt und ist die Grundlage der Anwendbarkeit der vorliegenden Lösung. Berechnete Glieder erster Ordnung des Störungsansatzes werden für alle vier Strömungen angegeben. Mehrere interessante Auswirkungen variabler Zähigkeit auf Strömung und Transporteigenschaften werden durch die vorliegenden Ergebnisse aufgezeigt. Diese Abweichungen von einer Lösung für konstante Zähigkeit erweisen sich sogar innerhalb des notwendigerweise begrenzten Gültigkeitsbereichs einer Lösung durch Störungsansatz erster Ordnung als wesentlich. Die Ergebnisse für den Wärmeübergang bei isothermen Oberflächen und bei solchen mit konstantem Wärmestrom befinden sich in sehr guter Übereinstimmung mit den entsprechenden Daten und Korrelationen vorausgegangener Untersuchungen. Die vorliegenden Ergebnisse stellen einige vorausgegangene Schlüsse bezüglich Auftriebsströmungen in einen klareren Zusammenhang.

ВЛИЯНИЕ ПЕРЕМЕННОЙ ВЯЗКОСТИ НА НЕКОТОРЫЕ ТИПЫ СВОБОДНОКОНВЕКТИВНЫХ ТЕЧЕНИЙ

Аннотация — Методом возмущений анализируются три типа ламинарных свободноконвективных течений в жидкостях, вязкость которых зависит от температуры: свободно всплывающая плоская струя, восходящий поток над горизонтальным линейным источником около адиабатической поверхности (пристенная плоская струя) и течение возле вертикальной однородно нагреваемой поверхности. Хотя для этих течений имеются известные степенные автомодельные решения при постоянной вязкости, однако автомодельность исчезает, если вязкость рассматривается как функция температуры. Единственный тип течения, для которого сохраняется автомодельность течение возле вертикальной изотермической поверхности, включен в анализ с целью оценки степени применимости метода возмущений. Используемый в работе подход обеспечивает возможность единого способа учёта влияния переменной вязкости на указанные типы течения. Именно вязкость из всех свойств обычных жидкостей (за исключением воды) наиболее сильно зависит от температуры. Этот общепризнанный факт использовался при описании других течений и был положен в основу настоящего анализа. Рассчитанные величины в первом порядке по возмущению представлены для всех рассматриваемых типов течения. Получены некоторые интересные выводы о влиянии переменной вязкости на течение и перенос тепла, которые представляются существенными даже с учётом ограниченности применимости результатов, полученных в первом порядке по возмущению. Результаты по теплообмену для изотермической поверхности и поверхности с постоянным тепловым потоком находятся в хорошем согласии с соответствующими данными предыдущих исследований. Полученные результаты также уточняют некоторые ранее сделанные выводы относительно подъёмных течений.